ASYMPTOTIC FORMS OF COULOMB WAVE FUNCTIONS, I

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With an Appendix by

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1. Introduction

Coulomb wave functions are solutions of the differential equation [15]

$$(1.1)\quad \frac{d^2y}{d\rho^2} + \left(1 - \frac{2\eta}{\rho} - \frac{L(L+1)}{\rho^2}\right)y = 0$$

where L is a real constant (integer in the case of non-relativistic Coulomb wave functions) which we may take $\geq -\frac{1}{2}$, η is a positive parameter, and ρ is a real variable, $0 < \rho < \infty$. The substitution*

$$(1.2)\quad z = 2i\rho, \quad k = i\eta, \quad m = L + \tfrac{1}{2}$$

reduces (1.1) to Whittaker's confluent hypergeometric equation

$$(1.3)\quad \frac{d^2w}{dz^2} + \left(-\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2}\right)w = 0,$$

and many analytic properties of Coulomb wave functions follow from the theory of confluent hypergeometric functions [16], [8].

It follows from the theory of confluent hypergeometric functions that for $L \geq -\frac{1}{2}$ there exists a solution of (1.1) which behaves like ρ^{L+1} near $\rho = 0$, and also that upon normalization of this solution, F_L, according to

$$(1.4)\quad F_L(\eta, \rho) = C_L(\eta)\,\rho^{L+1}\,[1 + O(\rho)] \qquad \text{as} \quad \rho \to 0$$

where

$$(1.5)\quad C_L(\eta) = 2^L\, e^{-\frac{1}{2}\pi\eta}\, |\Gamma(L + 1 + i\eta)|/\Gamma(2L+2),$$

the behavior for large ρ is given by

$$(1.6)\quad F_L(\eta, \rho) = \sin[\rho - \eta\log(2\rho) - \tfrac{1}{2}L\pi + \sigma_L(\eta)] + O(\rho^{-1})$$

as $\rho \to \infty$, where

$$(1.7)\quad \sigma_L(\eta) = \arg\Gamma(L + 1 + i\eta).$$

* The symbol m used here must not be confused with the integer denoted by m in the later sections of this report.

A second solution, G_L, of (1.1) can then be defined uniquely by the condition

(1.8) $$G_L(\eta, \rho) = \cos[\rho - \eta\log(2\rho) - \tfrac{1}{2}L\pi + \sigma_L(\eta)] + O(\rho^{-1})$$

as $\rho \to \infty$: this solution is such that $\rho^{-\frac{1}{2}} G_L$ is unbounded at $\rho = 0$, in fact $G_L = O(\rho^{-L})$ as $\rho \to 0$, except when $L = -\frac{1}{2}$ when $\rho^{-\frac{1}{2}} G_L = O(\log\rho)$ as $\rho \to 0$. The two solutions defined by (1.6) and (1.8) are known, respectively, as the regular and irregular Coulomb wave functions.

In this report we propose to obtain approximations for both regular and irregular Coulomb wave functions and their derivatives when the parameter η is large and the variable ρ is unrestricted. The differential equation (1.1) has a transition point at $\rho = 2\eta$: to the right of this transition point all solutions of (1.1) are oscillatory while to the left of this point the solutions show a monotonic behavior. Around $\rho = 2\eta$ is a transition region in which every solution changes from a monotonic to an oscillatory behavior. In order to obtain uniform asymptotic forms for unbounded intervals of ρ which include the transition point, we use the methods developed by Langer [12] and Cherry [9], and compare (1.1) with the Airy equation. Following Cherry, $\rho = \infty$ is chosen as the point of comparison, although the nature of the singularity at $\rho = \infty$ precludes a straightforward application of the theory developed by Cherry. The results presented here are incomplete in one respect: our approximations for regular Coulomb wave functions and their derivatives are not satisfactory when $\rho < 2\eta$. We hope to provide approximations in this range in a sequel to the present report.

Asymptotic forms of Coulomb wave functions have been obtained by many authors. The work on asymptotic forms of confluent hypergeometric functions is summarized in Buchholz's book ([8], Chapter IV) and in a paper by Chang, Chu and O'Brien, [10]; both the method of steepest descents and Langer's method have been used in this work. Unfortunately, none of the results apply to the case investigated here. Of recent work on Coulomb wave functions we mention the contributions of Abramowitz and Antosiewicz, Abramowitz and Rabinowitz, Barfield and Broyles, Biedenharn et al., Breit and Hull, Newton, and Yost et al. These authors obtained numerous useful approximations and asymptotic forms of Coulomb wave functions but as far as we know our formulas are the first ones to be valid for unrestricted values of ρ. Moreover, most of the previous work was based on integral representations of Coulomb wave functions while we obtain our results from the differential equation by methods which we hope will be found useful with other similar differential equations.

2. Outline of the method

It will be convenient to introduce new parameters ν, a and a new independent variable x by means of the substitution

$$(2.1)\quad 2\eta = \nu, \qquad L(L+1) = a, \qquad \rho = \nu(1+x),$$

which carries (1.1) into

$$(2.2)\quad \frac{d^2y}{dx^2} + \nu^2\left[\frac{x}{1+x} - \frac{a}{\nu^2(1+x)^2}\right]y = 0,$$

where $a \geq -\frac{1}{4}$ is a fixed real parameter, ν is a large real parameter, $\nu \to +\infty$, and x is a real variable, $-1 < x < \infty$. The coefficient of y in (2.2) vanishes at

$$(2.3)\quad x_0 = \frac{1}{2}\left[-1 + \left(1 + \frac{4a}{\nu^2}\right)^{\frac{1}{2}}\right],$$

this coefficient being positive when $x > x_0$, and negative when $x < x_0$. Accordingly, the solutions of (2.2) show an oscillatory behavior when $x > x_0$, and a monotonic behavior when $x < x_0$. As ν increases indefinitely, $x_0 \to 0$, and we call $x = 0$ the *transition point* of (2.2).

The solutions of (2.2) will be compared with those of

$$(2.4)\quad \frac{d^2U}{dt^2} + \nu^2 t U = 0.$$

This last equation is the simplest known equation with a transition point (at $t = 0$), and it can be reduced easily to Airy's equation [13]. Denoting the well-known Airy functions by $Ai(z)$ and $Bi(z)$, and setting

$$(2.5)\quad \omega = \exp\frac{2\pi i}{3},$$

we see that for any integer r,

$$Ai(-\nu^{2/3}\, t\, \omega^r)$$

is a solution of (2.4), any two consecutive values of r giving rise to linearly independent solutions.

The two differential equations, (2.2) and (2.4), will be compared by means of a substitution of the form

$$(2.6)\quad t = \phi(x), \qquad U(t) = [\phi'(x)]^{\frac{1}{2}}\, Y(x)$$

which transforms (2.4) into

$$(2.7)\quad \frac{d^2 Y}{dx^2} + \nu^2 \left[\phi(x)\,\phi'^{\,2}(x) + \frac{1}{2\nu^2}\,\{\phi, x\} \right] Y = 0,$$

where

$$(2.8)\quad \{\phi, x\} = \frac{\phi'''(x)}{\phi'(x)} - \frac{3}{2}\left[\frac{\phi''(x)}{\phi'(x)}\right]^2$$

is the Schwarzian derivative of ϕ.

If it were possible to determine ϕ so as to make (2.2) and (2.7) identical then we should have succeeded in expressing Coulomb wave functions in terms of Airy functions exactly. An approximate expression of Coulomb wave functions in terms of Airy functions may be expected to emerge if ϕ is chosen so that (2.2) and (2.7) coincide approximately when ν is large. This will be achieved by choosing ϕ as a solution of

$$(2.9)\quad \phi(x)\,\phi'^{\,2}(x) = \frac{x}{1+x},$$

so that (2.7) becomes

$$(2.10)\quad \frac{d^2 Y}{dx^2} + \nu^2 \left[\frac{x}{1+x} + \frac{1}{2\nu^2}\{\phi, x\} \right] Y = 0$$

while (2.2) may be written as

$$(2.11)\quad \frac{d^2 y}{dx^2} + \nu^2 \left[\frac{x}{1+x} + \frac{1}{2\nu^2}\{\phi, x\} \right] y = G(x) y,$$

where

$$(2.12)\quad G(x) = \frac{1}{2}\{\phi, x\} + \frac{a}{(1+x)^2}\,.$$

Let

$$(2.13)\quad Y_m(x) = [\phi'(x)]^{-1/2}\, Ai(-\nu^{2/3}\,\omega^m\,\phi(x))$$

where m is an integer and $\omega = e^{2\pi i/3}$. Then Y_m is a solution of (2.10), consecutive values of m corresponding to linearly independent solutions. The solutions of (2.11) are compared with those of (2.10) by means of the integral equation

$$(2.14)\quad y(x) = Y(x) + \int_x^\infty K(x, t)\, G(t)\, y(t)\, dt$$

where

$$(2.15)\quad K(x, t) = \frac{1}{\Delta_m}\,[Y_m(x)\, Y_{m-1}(t) - Y_m(t)\, Y_{m-1}(x)]$$

and

$$(2.16)\quad \Delta_m = Y_m(z)\, Y'_{m-1}(z) - Y'_m(z)\, Y_{m-1}(z) = -\frac{\nu^{2/3}\,\omega^{m+1/4}}{2\pi}.$$

The function K does not depend on m. If Y is a solution of (2.10) and y is a solution of (2.14) then y is a solution of (2.11). It will be shown that when $Y = Y_1$ or $Y = Y_{-1}$ equation (2.14) has a solution on $-1 < x < \infty$ with the property that the integral term in (2.14) is small compared to either of the other terms.

3. The transformation $\phi(x)$.

Let

$$(3.1)\quad f(x) = \int_0^x \left(\frac{t}{1+t}\right)^{1/2} dt$$

where the integrand is taken to be positive on the positive real axis, and is defined on $-1 < x < 0$ by analytic continuation about the origin in the upper half-plane. The integral (3.1) can be evaluated in terms of elementary functions. Its value is given by

$$(3.2)\quad f(x) = [x(1+x)]^{1/2} - \log[(1+x)^{1/2} + x^{1/2}]$$

where the positive values of the square roots and of the logarithm are taken for positive x.

The function $\phi(x)$ defined by

$$(3.3)\quad \phi(x) = \left[\frac{3}{2} f(x)\right]^{2/3}$$

where that determination of the fractional power is taken which makes $\phi(x)$ real, is a solution of (2.9).

About the origin $f(x)$, $\phi(x)$ can be expanded as follows

$$(3.4)\quad f(x) = \frac{2}{3} x^{3/2} \left(1 - \frac{3}{10} x + \frac{9}{56} x^2 \ldots\right)$$

$$(3.5)\quad \phi(x) = x - \frac{1}{5}x^2 + \frac{17}{175}x^3 \cdots .$$

From (3.2) it is found that as $x \to \infty$

$$(3.6)\quad f(x) = x - \frac{1}{2}\log x + \left(\frac{1}{2} - \log 2\right) + O\left(\frac{1}{x}\right).$$

Hence,

$$(3.7)\quad \frac{1}{f^2(x)} = \frac{1}{x^2}[1 + o(1)] \quad \text{as} \quad x \to \infty.$$

$\phi(x)$ has a simple zero at the origin and, by (2.9), it follows that $\phi'(x) \neq 0$, $-1 < x < \infty$. Hence $\{\phi, x\}$ is an analytic function on $-1 < x < \infty$.

By direct calculation

$$(3.8)\quad \{f, x\} = \frac{3}{8(1+x)^2} - \frac{5}{8x^2} + \frac{1}{4x(1+x)} .$$

Substituting (3.1) and (3.8) in the identity

$$(3.9)\quad \{\phi, x\} = \{\phi, f\}\,(f'(x))^2 + \{f, x\}$$

it is found that

$$(3.10)\quad \{\phi, x\} = \frac{5}{18f^2}\,\frac{x}{(1+x)} + \frac{3}{8(1+x)^2} - \frac{5}{8x^2} + \frac{1}{4x(1+x)} .$$

From (3.7) and (3.10) it is clear that

$$(3.11)\quad \{\phi, x\} = O\left(\frac{1}{x^2}\right) \quad \text{as} \quad x \to \infty.$$

From (3.2) and (3.10)

$$(3.12)\quad \{\phi, x\} = O\left(\frac{1}{(1+x)^2}\right) \quad \text{as} \quad x \to -1.$$

Hence the function $G(x)$ of (2.12) is analytic on $-1 < x < \infty$ and

$$(3.13)\quad G(x) = O\left(\frac{1}{x^2}\right) \quad \text{as} \quad x \to \infty$$

$$= O\left(\frac{1}{(1+x)^2}\right) \quad \text{as} \quad x \to -1.$$

The section is concluded by a more precise examination of bounds for the transformation $\phi(x)$.

From (3.3) it follows that

$$(3.14)\quad \left[\frac{\phi(x)}{x}\right]^{3/2} = \frac{3}{2}\,\frac{1}{x(1+x)^{1/2}} \int_0^x \left(\frac{t}{x}\;\frac{1+x}{1+t}\right)^{1/2} dt.$$

Let

$$(3.15)\quad g(x) = \int_0^x \left(\frac{t}{x}\;\frac{1+x}{1+t}\right)^{1/2} dt.$$

It should be observed that the expression under the square root in the integrand is positive on $-1 < x < \infty$.

If $\quad 0 \le t \le x,\quad$ then $\dfrac{t}{1+t} \le \dfrac{x}{1+x}$ and $0 \le g(x) \le x$;

if $\quad -1 < x \le t \le 0,\quad$ then $\dfrac{-t}{1+t} \le \dfrac{-t}{1+x}$ and $0 \le -g(x) \le -\dfrac{2}{3}x$;

and if $\quad 0 \le t \le x,\quad$ then $\dfrac{t}{1+t} \ge \dfrac{t}{1+x}$ and $g(x) \ge \dfrac{2}{3}x$.

Hence

$$(3.16)\quad \operatorname{Min}\left(1, \frac{1}{(1+x)^{1/2}}\right) \le \left[\frac{\phi(x)}{x}\right]^{3/2} \le \frac{3}{2(1+x)^{1/2}} \quad \text{if } -1 < x < \infty,$$

and

$$\left[\frac{\phi(x)}{x}\right]^{3/2} \le (1+x)^{-1/2} \qquad \text{if } -1 < x \le 0.$$

4. The integral equation

For $m = -1, 1$, let

$$(4.1) \quad \psi_m(x) = \frac{y_m(x)}{Y_m(x)}$$

($m=0$ is excluded since $Y_0(x)$ has zeros for real positive x). The integral equation may be written as

$$(4.2) \quad \psi_m(x) = 1 + \int_x^\infty R_m(x, t)\, G(t)\, \psi_m(t)\, dt$$

where

$$(4.3) \quad R_m(x; t) = \frac{-2\pi}{\nu^{2/3}\, \omega^{m+1/4}} \left[Y_m(t)\, Y_{m-1}(t) - Y_m^2(t) \frac{Y_{m-1}(x)}{Y_m(x)} \right]$$

If a is any positive number and $-\pi \le x \le \pi$ then

$$(a + |z|)^{1/4} \exp\left(\frac{2}{3}\, z^{3/2}\right) Ai(z)$$

is bounded and is bounded away from zero if the zeros of $Ai(z)$ are suitably excluded. As will be explained in sec. 7, it is convenient to choose $a = 1/4$. Moreover since the bounds required here are those for constant value of arg z, this result is expressed as follows:

$$(4.4) \quad m(\theta) \le \left| (1/4 + |z|)^{1/4} \exp\left(\frac{2}{3}\, z^{3/2}\right) Ai(z) \right| \le M(\theta)$$

$$\theta = \arg z, \qquad -\pi \le \theta \le \pi,$$

where $M(\theta)$ is a bounded function of θ on $|\theta| \le \pi$ and $m(\theta)$ is bounded away from zero on $|\theta| \le \pi - \epsilon$ where ϵ is any positive number. In (4.4), $z^{3/2}$ has that determination obtained by continuation from real positive values on the real positive axis into the sector $-\pi \le \theta \le \pi$.

It follows from (2.13), (3.3), (4.3), and (4.4) that

$$(4.5) \quad |R_1(x, t)| \le \frac{2\pi}{\nu^{2/3}} \quad \frac{1}{\phi'(t)\,(1/4 + \nu^{2/3}\, |\phi(t)|)^{1/2}}\, K$$

where

(a) $K = A_1 = M(\pi)\, M(-\pi/3) + M^2(-\pi/3)\ \dfrac{M(\pi)}{m(-\pi/3)}$ if $0 \le x \le t$,

(b) $K = A_2 = M(\pi)\, M(-\pi/3) + M^2(-\pi/3)\ \dfrac{M(0)}{m(2\pi/3)}$ if $x < 0 \le t$,

(c) $K = A_3 = M(0)\, M(2\pi/3) + M^2(2\pi/3)\ \dfrac{M(0)}{m(2\pi/3)}$ if $x \le t < 0$.

According to (3.13) there is a constant B such that

$$(4.6)\quad |G(x)| \le \frac{B}{(1+x)^2} \qquad -1 < x < \infty.$$

If $t \ge x > -1$, then

$$(4.7)\quad |R_1(x, t)\, G(t)| \le \frac{2\pi}{\nu^{2/3}}\ \frac{AB}{\phi'(t)\,(\tfrac{1}{4} + \nu^{2/3}\,|\phi(t)|)^{1/2}\,(1+t)^2}$$

where

$$A = \max(A_1, A_2, A_3).$$

If $x \ge 0$ it is permitted to take $A = A_1$.

A solution of (4.2) on the interval $-1 < x < \infty$ can now be constructed by the method of successive approximations. The case $m = 1$ will be considered first. If

$$(4.8)\quad \begin{cases} \psi_1^{(0)}(x) = 1 \\ \psi_1^{(n)}(x) = \int_x^\infty R_1(x, t)\, G(t)\, \psi_1^{(n-1)}(t)\, dt & n \ge 1 \end{cases}$$

then the series

$$(4.9)\quad \psi_1(x) = \sum_{n=0}^{\infty} \psi_1^{(n)}(x)$$

defines a formal solution of the integral equation. (4.7) implies the weaker inequality

$$(4.10)\quad |R_1(x, t)\, G(t)| \le \frac{4\pi}{\nu^{2/3}}\; AB\, \frac{1}{\phi'(t)\,(1+t)^2}$$

$$= \frac{4\pi AB}{\nu^{2/3}} \left[\frac{\phi(t)}{t}\right]^{1/2} (1+t)^{-3/2}$$

for $-1 < x \le t < \infty$. Hence by virtue of (3.16)

$$(4.11)\quad |R_1(x, t)\, G(t)| \le \frac{4\pi AB}{\nu^{2/3}} \left(\frac{3}{2}\right)^{1/3} (1+t)^{-5/3} \quad \text{for } -1 < x \le t < \infty.$$

It follows from (4.8) and (4.11) by induction on n that

$$(4.12)\quad |\psi_1^{(n)}(x)| \le \frac{1}{n!}\left[\frac{C}{\nu^{2/3}\,(1+x)^{2/3}}\right]^n \qquad n = 0, 1, 2, \dots, x > -1$$

where

$$(4.13)\quad C = 4\pi AB\left(\frac{3}{2}\right)^{4/3}.$$

Consequently, the series (4.9) converges on $-1 < x < \infty$ to a solution of (4.2).

From (4.12) it follows that

$$(4.14)\quad |\psi_1(x) - 1| \le \exp\left[\frac{C}{(\nu(1+x))^{2/3}}\right] - 1 \qquad -1 < x < \infty.$$

This result is based on (4.10) which is a weak form of the inequality (4.7). An alternative weak form is given by

$$(4.15)\quad |R_1(x, t)\, G(t)| \le \frac{2\pi}{\nu^{2/3}}\, \frac{AB}{\phi'(t)\,\nu^{1/3}\,|\phi(t)|^{1/2}\,(1+t)^2} \quad -1 < x \le t.$$

Hence by (2.9)

$$(4.16)\quad |R_1(x, t)\, G(t)| \le \frac{2\pi AB}{\nu}\; \frac{1}{|t^{1/2}\,(1+t)^{3/2}|}.$$

It follows from (4.8) and (4.16) by induction on n that

$$(4.17)\quad |\psi_1^{(n)}(x)| \le \left(\frac{2\pi AB}{\nu}\, C_\epsilon\right)^n \qquad x \ge -1 + \epsilon,$$

where

$$(4.18)\quad C_\epsilon = \int_{-1+\epsilon}^{\infty} |t|^{-1/2}\,(1+t)^{-3/2}\,dt$$

$$= \begin{cases} 2\left(1+\left(\dfrac{1-\epsilon}{\epsilon}\right)^{1/2}\right) & \text{if} \quad 0<\epsilon\le 1, \\[2ex] 2\left(1-\left(\dfrac{\epsilon-1}{\epsilon}\right)^{1/2}\right) & \text{if} \quad \epsilon > 1. \end{cases}$$

From (4.17) it follows that

$$(4.19)\quad |\psi_1(x)-1| \le \frac{\dfrac{2\pi AB}{\nu}\,C_\epsilon}{1-\dfrac{2\pi AB}{\nu}C_\epsilon} \qquad \text{if} \quad x\ge -1+\epsilon, \quad \nu > 2\pi ABC_\epsilon,$$

and hence

$$(4.20)\quad \psi_1(x) = 1 + O(1/\nu) \qquad x\ge -1+\epsilon, \quad \nu\ge\nu_0$$

where $\nu_0 > 2\pi ABC_\epsilon$.

A stronger result may be obtained for positive x if in (4.17) C_ϵ is replaced by

$$(4.21)\quad C_x = \int_x^{\infty} |t|^{-1/2}\,(1+t)^{-3/2}\,dt = 2\left[1-\left(\frac{x}{1+x}\right)^{1/2}\right] = O\left(\frac{1}{x}\right).$$

Since this replacement is permissible there is a positive number k such that

$$(4.22)\quad \psi_1(x) = 1 + O\left(\frac{1}{\nu x}\right) \qquad \nu x \ge k.$$

The results for $m=-1$ follow easily. Observing that

$$R_{-1}(x,t) = \frac{-2\pi}{\nu^{2/3}\,\omega^{-3/4}}\left[Y_{-1}(t)\,Y_1(t) - Y_{-1}^2(t)\,\frac{Y_1(x)}{Y_{-1}(x)}\right],$$

and using the well-known relation between solutions of the Airy equation,

$$Ai(z) + \omega Ai(\omega z) + \omega^2 Ai(\omega^2 z) = 0$$

gives

$$(4.23)\quad R_{-1}(x, t) = \frac{-2\pi}{\nu^{2/3}}\, \omega^{5/4} \left(Y_0(t)\, Y_{-1}(t) - \frac{Y_0(x)}{Y_{-1}(x)}\, Y_{-1}^2(t) \right).$$

Since

$$\overline{Y}_1(t) = Y_{-1}(t),$$

it follows that

$$R_{-1}(x, t) = \overline{R_1(x, t)},$$

the bars denoting complex conjugation.

Hence $\psi_{-1}(x) = \overline{\psi}_1(x)$ is a solution of the integral equation (4.2) for $m = -1$. Thus (4.14), (4.19) and (4.20) give rise to corresponding results with the suffix 1 replaced by -1.

Asymptotic representations for two solutions ψ_1, ψ_{-1} of the integral equation (4.2) have now been found. The functions

$$y_m(x) = Y_m(x)\, \psi_m(x) \qquad m = 1, -1,$$

are (conjugate complex) solutions of (2.14) and hence of (2.2). These solutions have the properties

$$(4.24)\quad |y_m(x) - Y_m(x)| \le |Y_m(x)| \left[-1 + \exp \frac{C}{[\nu(1+x)]^{2/3}} \right]$$

for $-1 < x < \infty$, $\nu > 0$, the constant C being given by (4.13);

$$(4.25)\quad y_m(x) = Y_m(x)\, [1 + O(1/\nu)]$$

for $x \ge -1 + \epsilon$, $\nu > \nu_0$, where ν_0 and the constant in the O-symbol are determined as in (4.20);

$$(4.26)\quad y_m(x) = Y_m(x)\, [1 + O(1/\nu x)]$$

for $\nu x \ge k$, where k and the constant in the O-symbol are determined as in (4.22).

5. Asymptotic forms for $y_1'(x)$, $y_{-1}'(x)$.

Let

(5.1) $z_n(x) = [\phi'(x)]^{1/2}\, y_n(x)$

and

(5.2) $Z_n(x) = [\phi'(x)]^{1/2}\, Y_n(x) = Ai(-\nu^{2/3}\,\phi(x)\,\omega^n)$.

From (2.14), (2.15), (2.16) and (4.1) it follows that

$$(5.3)\quad \frac{z_1'(x)}{Z_1'(x)} = 1 - \frac{2\pi}{\nu^{2/3}\,\omega^{5/4}} \int_x^\infty \left[Y_0(t)\,Y_1(t) - Y_1^2(t)\,\frac{Z_0'(x)}{Z_1'(x)} \right] G(t)\,\psi_1(t)\,dt$$

$$= 1 + \int R_1'(x, t)\, G(t)\, \psi_1(t)\, dt$$

where

$$(5.4)\quad R_1'(x, t) = \frac{-2\pi}{\nu^{2/3}\,\omega^{5/4}} \left[Y_0(t)\,Y_1(t) - Y_1^2(t)\,\frac{Z_0'(x)}{Z_1'(x)} \right].$$

As in sec. 4, it follows from the properties of Airy functions that

$$(5.5)\quad m'(\theta) \le \left| (1+|z|)^{-1/4} \exp\left(\frac{2}{3}\; z^{3/2}\right) Ai'(z) \right| \le M'(\theta);$$

$$\theta = \arg z,\ -\pi \le \theta \le \pi$$

where $M'(\theta)$ is a bounded function of θ and $m'(\theta)$ is bounded away from zero if $|\theta| \le \pi - \epsilon$ where ϵ is any positive number.

Hence as in sec. 4,

$$(5.6)\quad |R_1'(x, t)| \le \frac{2\pi}{\nu^{2/3}}\; \frac{1}{\phi'(t)\,[1/4 + \nu^{2/3}\,|\phi(t)|]^{1/2}}\, K'$$

where

(a) $K' = A_1' = M(\pi)\,M(-\pi/3) + M^2(-\pi/3)\;\dfrac{M'(\pi)}{m'(-\pi/3)}$ if $0 \le x \le t$

(b) $K' = A_2' = M(\pi)\,M(-\pi/3) + M^2(-\pi/3)\;\dfrac{M'(0)}{m'(2\pi/3)}$ if $x < 0 \le t$

(c) $K' = A_3' = M(0)\,M(2\pi/3) + M^2(2\pi/3)\,\dfrac{M'(0)}{m'(2\pi/3)}$ if $x \leq t < 0$

Let $A' = \max(A_1', A_2', A_3')$. A weak form of the inequality (5.6) is given by

$$(5.7)\quad |R_1'(x,t)| \leq \frac{2\pi}{\nu}\,\frac{A'}{|\phi(t)\,\phi'^2(t)|^{1/2}} \qquad -1 < x \leq t$$

Hence by (4.6), (5.3) and (5.7)

$$\left|\frac{z_1'(x)}{Z_1'(x)} - 1\right| \leq \frac{2\pi}{\nu} A'\,B \int_x^{\infty} |t|^{-1/2}\,(1+t)^{-3/2}\,|\psi_1(t)|\,dt$$

$$\leq \frac{2\pi A'\,B}{\nu}\,C_\epsilon \sup_{-1+\epsilon<t<\infty} |\psi_1(t)|\,.$$

From (4.19) it follows that

$$(5.8)\quad \left|\frac{z_1'(x)}{Z_1'(x)} - 1\right| \leq \frac{2\pi A'B}{\nu}\,C_\epsilon + \frac{(2\pi/\nu)^2\,AA'B^2C_\epsilon^2}{1 - \dfrac{2\pi AB}{\nu}\,C_\epsilon}$$

$$x \geq -1+\epsilon, \quad \nu > 2\pi ABC_\epsilon$$

and therefore

$$(5.9)\quad z_1'(x) = Z_1'(x)\,[1 + O(1/\nu)] \qquad x \geq -1+\epsilon, \quad \nu \geq \nu_0$$

Again as in sec. 3 a stronger result may be obtained for positive x by replacing C_ϵ by C_x in (5.8) to obtain

$$(5.10)\quad z_1'(x) = Z_1'(x)\,[1 + O(1/\nu x)] \qquad \nu x \geq k.$$

Corresponding results are obtained for $z_{-1}(x)$ by replacing the suffix 1 by -1 in formulas (5.8), (5.9) and (5.10). Thus for $m = 1, -1$,

$$
\begin{aligned}
y_m'(x) &= [\phi'(x)]^{-1/2}\, z_m'(x) - \frac{1}{2}\frac{\phi''(x)}{\phi'(x)}\, y_m(x) \\
&= [\phi'(x)]^{-1/2}\, Z_m'(x)\left[1 + O\left(\frac{1}{\nu}\right)\right] - \frac{1}{2}\frac{\phi''(x)}{\phi'(x)}\, Y_m(x)\left[1 + O\left(\frac{1}{\nu}\right)\right] \\
&= Y_m'(x)\left[1 + O\left(\frac{1}{\nu}\right)\right] + \frac{1}{2}\frac{\phi''(x)}{\phi'(x)}\, Y_m(x)\, O\left(\frac{1}{\nu}\right)
\end{aligned}
$$

From (3.3) and (3.6) it is clear that

$$
\frac{\phi''(x)}{\phi'(x)} = O\left(\frac{1}{x}\right) \qquad \text{as} \qquad x \to \infty
$$

and since $\phi'(x) \neq 0$,

$$
\frac{\phi''(x)}{\phi'(x)} \quad \text{is bounded,} \qquad x \geq -1 + \epsilon.
$$

Hence for $m = -1, 1$

$$
(5.11) \quad y_m'(x) = Y_m'(x)\left[1 + O\left(\frac{1}{\nu}\right)\right] + Y_m(x)\, O\left(\frac{1}{\nu}\right)
$$

$$
x \geq -1 + \epsilon, \quad \nu \geq \nu_0
$$

Again a stronger form for positive x may be obtained. Thus

$$
(5.12) \quad y_m'(x) = Y_m'(x)\left[1 + O\left(\frac{1}{\nu x}\right)\right] + Y_m(x)\, O\left(\frac{1}{\nu x^2}\right)
$$

$$
\nu x \geq k > 0, \quad m = -1, 1.
$$

6. Asymptotic formulas for the standard Coulomb wave functions

In this section asymptotic formulas for the solutions $F_L(\eta, \rho)$ and $G_L(\eta, \rho)$ of (1.1) are found by expressing these solutions in terms of the solutions y_1 and y_{-1} of (2.2).

Let

$$
(6.1) \quad w(x) = G_L\left(\frac{\nu}{2}, \nu(1+x)\right) + iF_L\left(\frac{\nu}{2}, \nu(1+x)\right).
$$

From (1.6) and (1.8) it follows that

$$(6.2)\quad w(x) = \exp i\left[\nu(1+x) - \frac{\nu}{2}\log 2\nu(1+x) - \frac{L\pi}{2} + \sigma_L\right] + O\left(\frac{1}{\nu(1+x)}\right)$$

where $\sigma_L = \arg\Gamma(L + 1 + i\,\nu/2)$.

Let

$$(6.3)\quad \xi_1(x) = [\phi'(x)]^{-1/2} Ai(-\nu^{2/3}\phi(x)),$$

$$\xi_2(x) = [\phi'(x)]^{-1/2} Bi(-\nu^{2/3}\phi(x))$$

where Ai, Bi are the usual solutions of Airy's equation [13]. $Bi(z)$ is defined by

$$(6.4)\quad Bi(z) = i\{\omega^2 Ai(\omega^2 z) - \omega Ai(\omega z)\},$$

and it is real for real z. Since

$$Ai(z) + \omega Ai(\omega z) + \omega^2 Ai(\omega^2 z) = 0$$

it follows that

$$(6.5)\quad \xi_1(x) = -\omega Y_1(x) - \omega^2 Y_{-1}(x),$$

and (6.4) gives

$$(6.6)\quad \xi_2(x) = -i\omega Y_1(x) + i\omega^2 Y_{-1}(x).$$

Since ξ_1, ξ_2 are each the sum of complex conjugates, they are real. Let

$$(6.7)\quad \zeta_1(x) = -\omega y_1(x) - \omega^2 y_{-1}(x),$$

$$\zeta_2(x) = -i\omega y_1(x) + i\omega^2 y_{-1}(x).$$

ζ_1, ζ_2 are real functions of x.

Now $w(x)$, being a solution of (2.2), is a linear combination of y_1, y_{-1} or ζ_1, ζ_2. Therefore

$$(6.8)\quad w(x) = c_1\,\zeta_1(x) + c_2\,\zeta_2(x)$$

where c_1, c_2 are complex constants, which may be determined by keeping ν fixed and comparing the behavior of both sides of (6.8) as $x \to \infty$. This is done by making use of (6.2) and (4.26). The result is

(6.9) $c_1 = \pi^{1/2}\nu^{1/6} e^{i\theta(\nu)}, \qquad c_2 = -\pi^{1/2}\nu^{1/6} e^{i[\theta(\nu)+\pi/2]}$

where

(6.10) $\theta(\nu) = \frac{\nu}{2} + \frac{\nu}{2}\log\frac{2}{\nu} - \frac{L\pi}{2} + \frac{\pi}{4} + \arg\Gamma(L+1+i\nu/2).$

The real and imaginary parts of (6.8) give

(6.11) $G_L = \pi^{1/2}\nu^{1/6}\cos[\theta(\nu)]\,\zeta_1(x) + \pi^{1/2}\nu^{1/6}\sin[\theta(\nu)]\,\zeta_2(x)$

$$F_L = \pi^{1/2}\nu^{1/6}\sin[\theta(\nu)]\,\zeta_1(x) - \pi^{1/2}\nu^{1/6}\cos[\theta(\nu)]\,\zeta_2(x).$$

Now

$$\begin{aligned}\zeta_1 &= -\omega y_1 - \omega^2 y_{-1}\\ &= -\omega Y_1\psi_1 - \omega^2 Y_{-1}\psi_{-1}\\ &= \xi_1 - 2\,\mathrm{Re}[\omega Y_1(\psi_1 - 1)].\end{aligned}$$

Then substituting for Y_1 by means of (6.5) and (6.6),

(6.12) $\zeta_1(x) = \xi_1(x) + \xi_1(x)\,\mathrm{Re}[\psi_1(x) - 1] + \xi_2(x)\,\mathrm{Im}[\psi_1(x) - 1].$

Similarly,

(6.13) $\zeta_2(x) = \xi_2(x) + \xi_2(x)\,\mathrm{Re}[\psi_1(x) - 1] - \xi_1(x)\,\mathrm{Im}[\psi_1(x) - 1].$

Thus from (4.25), for $x \geq -1 + \epsilon$, $\nu \geq \nu_0$

(6.14)
$$\begin{aligned}G_L[\nu/2, \nu(1+x)] &= \pi^{1/2}\nu^{1/6}\{\xi_1(x)\cos[\theta(\nu)] + \xi_2(x)\sin[\theta(\nu)]\\ &\qquad + (\xi_1^2 + \xi_2^2)^{1/2}\,O(\nu^{-1})\}\\ F_L[\nu/2, \nu(1+x)] &= \pi^{1/2}\nu^{1/6}\{\xi_1(x)\sin[\theta(\nu)] - \xi_2(x)\cos[\theta(\nu)]\\ &\qquad + (\xi_1^2 + \xi_2^2)^{1/2}\,O(1/\nu)\}\end{aligned}$$

where $\theta(\nu)$ is given by (6.10) and ξ_1, ξ_2 are given by (6.3). The above formula for the regular function F_L is not useful when $-1 < x < 0$, since the leading term, which is a difference of two large quantities, then becomes of the same order as the error term.

According to (4.26) a stronger form of these results for $\nu x \geq k > 0$ may be obtained by replacing $O(1/\nu)$ by $O(1/\nu x)$ in formula (6.14). Hence for the solutions $F_L(\eta, \rho)$, $G_L(\eta, \rho)$ of (1.1)

$$G_L(\eta, \rho) = \pi^{1/2} (2\eta)^{1/6} \{q_1(\rho) \cos[\lambda(\eta)] + q_2(\rho) \sin[\lambda(\eta)] + [q_1^2(\rho) + q_2^2(\rho)]^{1/2} O(1/\eta)\}$$

(6.15)

$$F_L(\eta, \rho) = \pi^{1/2} (2\eta)^{1/6} \{q_1(\rho) \sin[\lambda(\eta)] - q_2(\rho) \cos[\lambda(\nu)] + [q_1^2(\rho) + q_2^2(\rho)]^{1/2} O(1/\eta)\}$$

where

$$(6.16) \qquad \lambda(\eta) = \eta - \eta \log \eta - \frac{L\pi}{2} + \frac{\pi}{4} + \arg \Gamma(L + 1 + i\eta)$$

and where

$$q_1(\rho) = \left[\phi'\left(\frac{\rho - 2\eta}{2\eta}\right) \right]^{-1/2} Ai\left[-\nu^{2/3} \phi\left(\frac{\rho - 2\eta}{2\eta}\right) \right]$$

(6.17)

$$q_2(\rho) = \left[\phi'\left(\frac{\rho - 2\eta}{2\eta}\right) \right]^{-1/2} Bi\left[-\nu^{2/3} \phi\left(\frac{\rho - 2\eta}{2\eta}\right) \right]$$

Again in (6.15), if $\rho - 2\eta \geq k > 0$ then $O(1/\eta)$ may be replaced by

$$O\left(\frac{1}{\rho - 2\eta}\right).$$

7. Estimation of the constants A and B

In this section use is made of J.C.P. Miller's Tables for the Airy integral [13], and the notation used here corresponds to that of the tables.

Equation (32) on page B9 of [13] reads

$$Ai(\omega z) = -\frac{1}{2} \omega^2 \{Ai(z) - i\,Bi(z)\}.$$

Therefore for real z,

$$|Ai(\omega z)| = \frac{1}{2} [Ai^2(z) + Bi^2(z)]^{1/2} = \frac{1}{2} F(z)[\sin^2 \chi(z) + \cos^2 \chi(z)]^{1/2}$$

where F and χ are chosen so that

$$Ai(z) = F(z) \sin \chi(z),$$

$$Bi(z) = F(z) \cos \chi(z).$$

Hence for real z,

$$(7.1) \quad |Ai(\omega z)| = \frac{1}{2} F(z).$$

Let

$$(7.2) \quad g(z) = \left| (\tfrac{1}{4} + |z|)^{\frac{1}{4}} \exp\left(\frac{2}{3} z^{3/2}\right) Ai(z) \right|.$$

In order to obtain A, it is necessary to calculate $M(0)$, $M(\pi)$, $M(-\pi/3)$, $m(-\pi/3)$, $M(2\pi/3)$, $m(-2\pi/3)$. $M(0)$ was estimated by plotting $g(r)$, $r > 0$ for $r = 0$, 0.25, 0.5, 0.75, 1, 1.5, 2.0 and at unit intervals to $r = 6$. The asymptotic value for $g(r)$ is known and the graph has an almost constant appearance ranging between 0.25 and 0.30.

Hence $M(0) \leq 0.3$.

For $M(\pi)$, $g(re^{i\pi})$ is considered. One has

$$|g(re^{i\pi})| \leq (\tfrac{1}{4} + r)^{\frac{1}{4}} Ai(-r) \leq (\tfrac{1}{4} + r)^{\frac{1}{4}} F(-r).$$

1 Again $(\tfrac{1}{4} + r)^{\frac{1}{4}} F(-r)$ was plotted at suitable intervals and it was found that

$$M(\pi) \leq 0.6.$$

For $M(-\pi/3)$, $m(-\pi/3)$, $g(re^{-i\pi/3})$ is considered. One has

$$\begin{aligned} |g(re^{-i\pi/3})| &= |(\tfrac{1}{4} + r)^{\frac{1}{4}} Ai(re^{-i\pi/3})| \\ &= |(\tfrac{1}{4} + r)^{\frac{1}{4}} Ai(-re^{2i\pi/3})| \\ &= \tfrac{1}{2}(\tfrac{1}{4} + r)^{\frac{1}{4}} F(-r) \qquad \text{by (7.1).} \end{aligned}$$

From the graph of $(\tfrac{1}{4} + r)^{\frac{1}{4}} F(-r)$ it was found that

$$M(-\pi/3) \leq 0.3, \qquad m(-\pi/3) \geq 0.25.$$

For $M(2\pi/3)$, $m(2\pi/3)$, $g(re^{2i\pi/3})$ is considered. One has

$$|g(re^{2i\pi/3})| = (\tfrac{1}{4} + r)^{1/4} \exp\left(-\frac{2}{3} r^{3/2}\right) |Ai(re^{2i\pi/3})|$$

$$= \tfrac{1}{2}(\tfrac{1}{4} + r)^{1/4} \exp\left(-\frac{2}{3} r^{3/2}\right) F(r) \qquad \text{by (7.1).}$$

By plotting this function for suitable intervals, it was found that

$$M(2\pi/3) \leq 0.35, \qquad m(2\pi/3) \geq 0.25.$$

Hence from (4.5) (a), (b), and (c) the constants A_1, A_2, A_3 can be bounded

$$A_1 \leq 0.4, \qquad A_2 \leq 0.3, \qquad A_3 \leq 0.3.$$

Hence a bound for $A = \max(A_1, A_2, A_3)$ is given by

$$A \leq 0.4.$$

Instead of working with the function

$$(\tfrac{1}{4} + |z|)^{1/4} \exp\left(\frac{2}{3} z^{3/2}\right) Ai(z)$$

in (4.4),

$$(a + |z|)^{1/4} \exp\left(\frac{2}{3} z^{3/2}\right) Ai(z)$$

or

$$(a + |z|^{1/4}) \exp\left(\frac{2}{3} z^{3/2}\right) Ai(z)$$

could have been used where $a > 0$. The form $g(z)$ was chosen so as to make the graphs drawn in this section as near to a constant as possible. This was done by a preliminary investigation, making use of the asymptotic form of $Ai(z)$.

To determine the constant B, an upper bound for $(1 + x)^2 \{\phi, x\}$ is obtained by plotting this function at suitable intervals, by means of formulas (3.2) and (3.10). It was found that

$$(1 + x)^2 \{\phi, x\} \leq 0.4$$

so that by (2.12)

$$|(1+x)^2 G(x)| \leq 0.2 + a.$$

Hence from (4.6)

$$B \leq 0.2 + a.$$

8. Appendix* on numerical results

It is the purpose of this section to tabulate numerical values of the asymptotic representations (6.15) and of the corresponding forms for the derivatives of Coulomb wave functions, and to compare the results with values given elsewhere [2, 3, 11, 15].

From (5.11), (6.5), (6.6), and (6.7) it follows that

$$(8.1) \quad \zeta_m'(x) \sim \xi_m'(x) \qquad x \geq -1+\epsilon, \quad \nu > \nu_0, \quad m = 1, 2,$$

and hence that (6.15) can be formally differentiated with respect to ρ. The error terms in (8.1) and in the subsequent asymptotic representations (8.2) and (8.4) have the same forms as the error terms in (6.12) and (6.15). The result of the formal differentiation is

$$\frac{dG_L(\eta,\rho)}{d\rho} \sim -\pi^{1/2}(2\eta)^{-1/6}\{\widetilde{q}_1(\rho)\cos[\lambda(\eta)] + \widetilde{q}_2(\rho)\sin[\lambda(\eta)]\}$$

(8.2)

$$\frac{dF_L(\eta,\rho)}{d\rho} \sim -\pi^{1/2}(2\eta)^{-1/6}\{\widetilde{q}_1(\rho)\sin[\lambda(\eta)] - \widetilde{q}_2(\rho)\cos[\lambda(\eta)]\},$$

where $\lambda(\eta)$ is given by (6.16) and where

$$\widetilde{q}_1(\rho) = \left[\phi'\left(\frac{\rho-2\eta}{2\eta}\right)\right]^{1/2} Ai'\left[-\nu^{2/3}\phi\left(\frac{\rho-2\eta}{2\eta}\right)\right]$$

(8.3)

$$\widetilde{q}_2(\rho) = \left[\phi'\left(\frac{\rho-2\eta}{2\eta}\right)\right]^{1/2} Bi'\left[-\nu^{2/3}\phi\left(\frac{\rho-2\eta}{2\eta}\right)\right].$$

* By C.A. Swanson.

The prime on the Airy integrals denotes differentiation with respect to the argument

$$-\nu^{2/3}\ \phi\left(\frac{\rho-2\eta}{2\eta}\right).$$

When (5.12) is used instead of (5.11), the order symbol $O(1/\nu)$ implied in (8.2) is replaced by $O[1/(\rho-2\eta)]$, $\rho-2\eta \geq K > 0$. For $L=0$ and $\rho = 2\eta$, equations (6.15) and (8.2) reduce to

$$
\begin{aligned}
&F_0(\eta, 2\eta) \sim \pi^{1/2}\,(2\eta)^{1/6}\{Ai(0)\sin[\lambda(\eta)] - Bi(0)\cos[\lambda(\eta)]\}\\
&G_0(\eta, 2\eta) \sim \pi^{1/2}\,(2\eta)^{1/6}\{Ai(0)\cos[\lambda(\eta)] + Bi(0)\sin[\lambda(\eta)]\}\\
&F_0'(\eta, 2\eta) \sim -\pi^{1/2}\,(2\eta)^{-1/6}\{Ai'(0)\sin[\lambda(\eta)] - Bi'(0)\cos[\lambda(\eta)]\}\\
&G_0'(\eta, 2\eta) \sim -\pi^{1/2}(2\eta)^{-1/6}\{Ai'(0)\cos[\lambda(\eta)] + Bi'(0)\sin[\lambda(\eta)]\},
\end{aligned}
\tag{8.4}
$$

where

$$\lambda(\eta) = \eta - \ln\eta + \frac{\pi}{4} + \arg\Gamma(1+i\eta).$$

For $\eta > 20$, the following formula is correct to 4 decimal places,

$$\lambda(\eta) = \frac{\pi}{2} - \frac{1}{12\eta}.$$

In the subsequent tabulation, the values of the logarithms, sines, and cosines were taken from the W.P.A. tables [19]. The fractional powers were obtained by iteration of square and cube roots. The values of the Airy integrals and of $\arg\Gamma(1+i\eta)$, $\eta \geq 20$, were taken from J. C. P. Miller's Table [13] and from Tables of Coulomb Wave Functions [15] respectively. The comparison values $F_0^T(\eta, 2\eta)$, $G_0^T(\eta, 2\eta)$, $F_0'^T(\eta, 2\eta)$, and $G'^T(\eta, 2\eta)$ were taken from the Abramowitz and Rabinowitz report [2] for $\eta \leq 25$, and from Barfield and Broyles [3] for $25 < \eta \leq 200$. Table I summarizes the results for the case $L=0$, $\rho = 2\eta$.

From Table I it is seen that the discrepancy between the values obtained from (8.4) and the comparison values is less than 0.5% for $\eta \geq 40$, less than 1.5% for $\eta \geq 10$, and less than 4% for $\eta \geq 3$. Since the comparison values $F_0^T(\eta, 2\eta)$ and $G_0^T(\eta, 2\eta)$ given by Barfield and Broyles are not reliable to three decimal places, the discrepancy in the third place for $\eta > 100$ is not necessarily interpreted as an absolute error.

Values of the asymptotic forms (6.15) off the transition line are tabulated in Table III for the values $L = 0$; $\eta = 3, 5$; $\rho = 1(1)10$. For this purpose, values of the function $\phi(x)$ satisfying (1.6) and the initial condition $\phi(0) = 0$ are given in the preliminary Table II. For $0 \leq x$. $\phi(x)$ is given by (3.3) and (3.4), while for $-1 \leq x \leq 0$,

$$\phi(x) = -\left[\frac{3}{2} f_*(x)\right]^{2/3}$$

where

$$f_*(x) = (-x^2 - x)^{½} - ½ \cos^{-1}(2x + 1) \qquad -1 \leq x \leq 0.$$

For convenience in tabulation, this is rewritten

$$f_*(x) = g(y) = [y(1 - y)]^{½} - \sin^{-1}(y^{½}) \qquad 0 \leq y = -x \leq 1.$$

The result

$$\phi'(x) = \left[\frac{1 + x}{x} \phi(x)\right]^{½}$$

was used in the computation of (6.17). The other functions involved in (6.15) and (6.17) were evaluated as in Table I. The comparison values of the Coulomb wave functions were obtained from the relations

$$F_0^T(\rho, \eta) = \rho f_{0,\eta}(\rho) C_0^{-1}(\eta) e^{-\pi\eta}$$

$$G_0^T(\rho, \eta) = \rho g_{0,\eta}(\rho) C_0^{-1}(\eta) e^{-\pi\eta},$$

where $f_{0,\eta}(\rho)$ and $g_{0,\eta}(\rho)$ are functions which are tabulated in a National Bureau of Standards report [11].

From Table III it is verified that the formulas (6.15) give good approximations for both $F_L(\eta, \rho)$ and $G_L(\eta, \rho)$ to the right of the transition point, even for η as small as 3, and that the formula for $G_L(\eta, \rho)$ gives a usable approximation in an interval including the transition point. However, the formula for $F_L(\eta, \rho)$ gives values which sharply decrease in accuracy to the left of the transition point. This inaccuracy is due to the cancellation of the dominant terms in the linear combination (6.15) when $\rho - 2\eta$ is negative. As noted in the introduction, the present results are then inadequate for describing $F_L(\eta, \rho)$ in an interval containing the transition point.

TABLE

TABULATION OF ASYMPTOTIC REPRESENTATIONS (8.4)

η	$F_0\ (\eta, 2\eta)$	$F_0^T\ (\eta, 2\eta)$	$G_0\ (\eta, 2\eta)$	$G_0^T\ (\eta, 2\eta)$
2.0	0.734	0.775	1.405	1.398
3.0	.807	.838	.492	.485
4.0	.857	.883	.560	.553
5.0	.870	.918	.615	.609
5.5	.914	.933	.640	.633
6.0	.929	.948	.662	.656
6.5	.943	.961	.684	.678
7.0	.957	.973	.704	.698
8.0	.981	.996	.741	.735
9.0	1.002	1.016	.774	.769
10.0	.022	.034	.805	.800
11	.039	.051	.832	.828
12	.056	.066	.859	.854
13	.071	.081	.883	.879
15	.098	.108	.927	.924
18	.134	.142	1.986	1.983
20	.155	.163	2.020	2.017
25	.201	.207	.096	.093
30	.239	.244	.160	.157
35	.272	.277	.216	.213
40	.301	.306	.265	.262
50	.352	.355	.350	.348
60	.394	.397	.422	.420
70	.431	.434	.485	.483
80	.463	.466	.541	.539
90	.492	.495	.591	.589
100	.519	.522	.637	.635
120	.567	.569	.718	.716
140	.608	.609	.789	.787
160	.644	.646	.851	.849
180	.677	.678	.908	.906
200	.707	.708	.959	.957

I

TO COULOMB WAVE FUNCTIONS, $L = 0$, $\rho = 2\eta$.

$F_0'(\eta, 2\eta)$	$F_0'^T(\eta, 2\eta)$	$G_0'(\eta, 2\eta)$	$F_0'^T(\eta, 2\eta)$
0.390	0.404	- 0.615	- 0.562
.357	.369	.579	.540
.336	.347	.555	.523
.322	.331	.536	.509
.316	.325	.528	.503
.310	.319	.521	.498
.306	.314	.514	.493
.302	.310	.508	.488
.294	.302	.497	.479
.288	.295+	.488	.471
.283	.290	.480	.464
.278	.284	.473	.458
.273	.279	.466	.452
.270	.275	.460	.447
.263	.268	.449	.438
.255	.259	.436	.426
.250	.254	.429	.420
.240	.244	.413	.406
.233	.235	.401	.396
.227	.229	.391	.386
.222	.224	.382	.378
.214	.215	.369	.366
.207	.208	.358	.355
.202	.203	.349	.346
.197	.198	.341	.339
.193	.194	.334	.333
.190	.191	.328	.327
.184	.185	.319	.318
.180	.180	.311	.310
.176	.176	.304	.303
.172	.172	.298	.297
.169	.169	.293	.292

+Discrepancy in tabulated values of $F_0'(\eta, 2\eta)$ in [2] and [3]:

TABLE II

VALUES OF $\phi(x)$ FOR $-1 \leq x \leq 10$

x	$\phi(x)$	x	$\phi(x)$	x	$\phi(x)$	x	$\phi(x)$	x	$\phi(x)$
-1.0	-1.7707	-0.3	-0.3212	0.4	0.3730	1.4	1.1553	4.5	3.0082
-0.9	-1.2655	-0.2	-0.2089	0.5	0.4593	1.6	1.2952	5.0	3.2648
-0.8	-1.0401	-0.1	-0.1021	0.6	0.5435	1.8	1.4312	6.0	3.7558
-0.7	-0.8613	0.0	0.0000	0.7	0.6255	2.0	1.5635	7.0	4.2219
-0.6	-0.7068	0.1	0.9809	0.8	0.7058	2.5	1.8804	8.0	4.6675
-0.5	-0.5680	0.2	0.1927	0.9	0.7843	3.0	2.1809	9.0	5.0956
-0.4	-0.4404	0.3	0.2842	1.0	0.8612	3.5	2.4677	10.0	5.5087
				1.2	1.0108	4.0	2.7429		

TABLE III

ASYMPTOTIC REPRESENTATIONS (6.15) TO COULOMB WAVE FUNCTIONS;

$L = 0;\ \eta = 3, 5;\ \rho = 1(1)10.$

ρ	$F_0(3,\rho)$	$F_0^T(3,\rho)$	$G_0(3,\rho)$	$G_0^T(3,\rho)$	$F_0(5,\rho)$	$F_0^T(5,\rho)$	$G_0(5,\rho)$	$G_0^T(5,\rho)$
1								
2								
3			5.896	6.019			169.4	173.5
4			3.116	3.144			48.5	49.4
5	0.440	0.489	2.077	2.079			17.92	18.19
6	0.807	0.838	1.492	1.485			8.18	8.27
7	1.162	1.180	0.929	0.913	0.100	0.174	4.51	4.55
8	1.349	1.354	0.246	0.228	0.300	0.345	2.943	2.952
9	1.202	1.198	-0.484	-0.501	0.570	0.600	2.153	2.151
10	0.670	0.660	-1.050	-1.060	0.897	0.918	1.615	1.608

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